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ABSTRACTS*

Jackson-type inequalities in the Musielak-Orlicz spaces

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Let

$$\mathbf{M} = \{M_k(u)\}_{k\in\mathbb{Z}}, \quad u \geq 0,$$

be a sequence of nondecreasing convex functions, $M_k(0) = 0$, $M_k(u) \to \infty$ as $u \to \infty$. The modular space (or Musilak-Orlicz space) $\mathcal{S}_\mathbf{M}$ is the space of 2π-periodic Lebesgue summable functions f, defined on the real axis such that the following quantity (the Orlicz norm of f) is finite:

$$\|f\|^*_\mathbf{M} := \sup\left\{\sum_{k\in\mathbb{Z}} \lambda_k |\widehat{f}(k)| : \quad \lambda_k \geq 0, \ \sum_{k\in\mathbb{Z}} M_k(\lambda_k) \leq 1\right\},$$

where

$$\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x$$

are the Fourier coefficients of the function f.

Consider the set Φ of all continuous bounded nonnegative pair functions φ such that $\varphi(0) = 0$ and the Lebesgue measure of the set $\{t \in \mathbb{R} : \varphi(t) = 0\}$ is equal to zero. For a fixed function $\varphi \in \Phi$ and for any $f \in \mathcal{S}_\mathbf{M}$, we define the generalized modulus of smoothness ω_φ of a function $f \in \mathcal{S}_\mathbf{M}$ by the equality:

$$\omega_\varphi(f,\delta)^*_\mathbf{M} := \sup_{|h|\leq\delta} \sup\Big\{\sum_{k\in\mathbb{Z}} \lambda_k |\varphi(kh)\widehat{f}(k)| : \quad \lambda_k \geq 0, \ \sum_{k\in\mathbb{Z}} M_k(\lambda_k) \leq 1\Big\}, \ \delta \geq 0.$$

Let $\mathcal{M}(\tau)$, $\tau > 0$, be a set of bounded nondecreasing functions μ that differ from a constant on $[0,\tau]$. By

$$\Omega_\varphi(f,\tau,\mu,u)^*_\mathbf{M}, \quad u > 0,$$

denote the average value of the generalized modulus of smoothness $\omega_\varphi(f,t)^*_\mathbf{M}$ of f with the weight $\mu \in \mathcal{M}(\tau)$, i.e.,

$$\Omega_\varphi(f,\tau,\mu,u)^*_\mathbf{M} := \frac{1}{\mu(\tau)-\mu(0)}\int_0^u \omega_\varphi(f,t)^*_\mathbf{M}\mathrm{d}\mu\Big(\frac{\tau t}{u}\Big).$$

For any function $f \in \mathcal{S}_\mathbf{M}$, denote by $E_n(f)^*_\mathbf{M}$ its best approximation by the trigonometric polynomials of the order $n-1$ in the space $\mathcal{S}_\mathbf{M}$.

Theorem. *Assume that*

$$f \in \mathcal{S}_\mathbf{M}, \quad \varphi \in \Phi, \quad \tau > 0, \quad \mu \in \mathcal{M}(\tau).$$

Then for any $n \in \mathbb{N}$

$$E_n(f)^*_\mathbf{M} \leq \frac{\mu(\tau)-\mu(0)}{I_{n,\varphi}(\tau,\mu)}\Omega_\varphi\Big(f,\tau,\mu,\frac{\tau}{n}\Big)^*_\mathbf{M}, \tag{1}$$

where

$$I_{n,\varphi}(\tau,\mu) := \inf_{k\in\mathbb{N}:k\geq n}\int_0^\tau \varphi\Big(\frac{kt}{n}\Big)\mathrm{d}\mu(t).$$

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If, in addition, φ is non-decreasing on $[0,\tau]$ and

$$I_{n,\varphi}(\tau,\mu)=\int_0^\tau \varphi(t)\mathrm{d}\mu(t),$$

then inequality (1) *can not be improved and therefore,*

$$\sup_{\substack{f\in\mathcal{S}_{\mathbf{M}}\\ f\neq\mathrm{const}}}\frac{E_n(f)^*_{\mathbf{M}}}{\Omega_\varphi(f,\tau,\mu,\frac{\tau}{n})^*_{\mathbf{M}}}=\frac{\mu(\tau)-\mu(0)}{\int_0^\tau\varphi(t)\mathrm{d}\mu(t)}.$$

Ulam stability for composition of operators

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The Ulam stability (see, e.g., [4]) for composition of operators was investigated in the framework of Fréchet spaces in [5], where the proofs had a "topological" character. Proofs with a "metric" character were given in [1] in the framework of Banach spaces. A C_0-semigroup $(T(t))_{t\geq 0}$ with each $T(t)$ Ulam stable was presented in [1]. The paper [2] is devoted to a C_0-semigroup (R_t) such that each R_t with $t>0$ is Ulam unstable. A crucial property was the injectivity of R_t, derived from the injectivity of the Weierstrass transform, see [3]. In this talk we give details concerning the above facts and add some new results. In particular, two open problems are presented.

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On approximation of ratios of multiple hypergeometric functions by branched continued fractions

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Continued fractions are effective tools for approximations of hypergeometric functions of one variable. In most cases, a given function can be represented by several different types of continued fractions, each with its own convergence domain [1, 2].

Multiple hypergeometric functions are a natural generalization of the hypergeometric functions of one variable [3]. Branched continued fractions (BCF) are used to construct rational approximations of functions of several variables, in particular, to approximations of the ratios of multiple hypergeometric functions [4]–[10]. In this case we need to solve the following problems:

- to construct the expansion of ratio of multiple hypergeometric functions into BCF;
- to investigate the convergence of this expansion;
- to prove that the BCF converges to a function which is an analytic continuation of the ratio of multiple hypergeometric functions in some domain.

In this talk we shall consider some approaches to solving these problems.

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On modified Mellin-Gauss-Weierstrass convolution operators

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This study is devoted to Mellin operators and their variants to improve approximation accuracy and approximate ratio. Two Mellin type operators are reconstructed by using two sequences of functions to enable better approximation. Keeping the idea of Mellin convolution, these classes aim to be associated with functions defined on the semi-real axis, and the afine and quadratic functions pairs are fixed. It has been shown, both theoretically and numerically, that the operators can be used to approximate function. Indeed the approximation accuracy can be adjusted by tuning the parameters. Moreover, weighted approximation, as well as Voronovskaya type results, are studied throughout the study. The advantages of each operator over the other in terms of both approximation errors and convergence rates are presented.

Approximation by discrete operators of Max-Product kind

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In this presentation, we investigate pseudo linear discrete operators, where the discrete kernel satisfies some suitable assumptions. We intuitively inspired by the researches [1–3]. Our aim is to prove the max-product approximation of these discrete operators for both univariate and multivariate cases. We also study the order of convergence. In the end, we illustrate our estimations for both cases with some kernels which fulfill our kernel assumptions.

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Approximation of some classes of L-space valued periodic functions by generalized trigonometric polynomials

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Let C and L_p, $1 \le p \le \infty$, be the spaces of 2π-periodic functions $f:\mathbb{R}\to\mathbb{R}$ with norms $\|\cdot\|_C$ and $\|\cdot\|_p$ respectively. Let also X be L_p, $1\le p\le\infty$, or C, and H be a finite-dimensional subspace of the space X. For $\mathcal{M}\subset X$ we set

$$E(\mathcal{M},H)_X = \sup_{f\in\mathcal{M}} \inf_{T\in H} \|f-T\|_X. \tag{1}$$

Quantity (1) is called the best approximation of the class $\mathcal{M}$ by the subspace H in the metric of the space X.

As usual, by $K*\varphi$ we denote the convolution of the functions $K\in L_p$ and $\varphi\in L_1$. For $1\le p\le\infty$ set

$$F_p = \{\varphi\in L_p\colon \|\varphi\|_p\le 1\} \quad \text{and} \quad K*F_p=\{K*\varphi\colon \varphi\in F_p\}.$$

It is well known that many important classes of numerical periodic functions are classes of the type $K*F_p$.

By H^T_{2n-1}, $n=1,2,\ldots$, we denote the set of trigonometric polynomials $T_{n-1}(x)$ of degree at most $n-1$.

We say that a kernel K satisfies the condition N_n^*, if there exist a polynomial $T^*\in H^T_{2n-1}$ and a point $\theta\in[0,\pi/n]$ such that $(K(x)-T^*(x))\varphi_n(x-\theta)\ge 0$ almost everywhere. Here and in what follows,

$$\varphi_n(x) := \operatorname{sgn}\sin nx.$$

Many kernels important for the approximation theory satisfy the condition N_n^*.

Theorem (S.M. Nikol'skii). *If a kernel K satisfies the condition N_n^*, then*

$$E(K * F_\infty, H_{2n-1}^T)_C = E(K * F_1, H_{2n-1}^T)_1 = E(K, H_{2n-1}^T)_1 = \|K * \varphi_n\|_C.$$

We generalize this and some other known results to the case of the best approximation of L-space valued functional classes (i. e. the range of the functions belongs to a semi-linear metric space with two additional axioms, which connect the metric with the algebraic operations) by generalized trigonometric polynomials. Consideration of L-space valued functions gives a unified approach to solutions of a series of extremal problems for the classes of multi- and fuzzy-valued functions, as well as for the classes of functions with values in Banach spaces, in particular random processes, and many other classes of functions.

Optimal recovery of monotone operators in partially ordered semi-linear metric spaces

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The concept of quasinormed space was introduced by S. Aseev in 1985. The axioms of a quasinormed space are satisfied for example by the space of closed bounded subsets of an arbitrary Banach space, as well as many spaces of fuzzy sets. We consider the problem of optimal recovery of monotone operators in the space of functions with values in quasinormed spaces. In particular, we obtain a wide generalization of Kiefer's well-known result on the optimal recovery of integrals of monotone functions defined on a finite interval.

The nonlocal boundary value problem with perturbations of mixed boundary conditions for an elliptic homogeneous equation with constant coefficients

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In this report we continue to investigate the properties of the problem with nonlocal conditions, which are multipoint perturbations of mixed boundary conditions, started in [1,2].

Let $G := \{x := (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1,\ x_2 < 1\}$, $D_1,\ D_2$ be the operators of differentiation by the variables $x_1,\ x_2$ respectively.

Let's consider the multipoint problem

$$L(D)u := \sum_{p=0}^{n} a_p D_1^{2p} D_2^{2n-2p} u = 0,\ x \in G, \tag{1}$$

$$\ell_{s,1} u := D_1^{2s-2} u|_{x_1=0} + D_1^{2s-2} u|_{x_1=1} = 0,\ s = 1, 2, ..., n, \tag{2}$$

$$\ell_{n+s,1}u := D_1^{2s-2}u|_{x_1=0} - D_1^{2s-2}u|_{x_1=1} = 0,\ s = 1, 2, ..., n, \quad (3)$$

$$\ell_{s,2}u := D_2^{2s-2}u|_{x_2=0} + D_2^{2s-2}u|_{x_2=1} = f_s(x_1),\ s = 1, 2, ..., n, \quad (4)$$

$$\ell_{n+s,2}u := D_2^{2s-1}u|_{x_2=0} + D_2^{2s-1}u|_{x_2=1} + \ell_s u = f_{n+s}(x_1),\ s = 1, 2, ..., n, \quad (5)$$

$$\ell_s u := \sum_{q=0}^{k_s}\sum_{r=1}^{m} b_{q,r,s} D_2^q u(x)|_{x_2=x_{2,r}},\ s = 1, 2, ..., n, \quad (6)$$

$$0 = x_{2,1} < x_{2,2} < ... < x_{2,k} = 1,\ a_p,\ b_{q,r,s} \in \mathbb{R},$$

$$q = 0, 1, ..., k_s,\ k_s < 2n,\ r = 0, 1, ..., m,\ s = 1, 2, ..., n,\ p = 0, 1, ..., n.$$

We construct the operator which maps the solutions of the self-adjoint boundary-value problem with mixed boundary conditions ($b_{q,r,s} = 0$) to the solutions of the investigated multipoint problem.

In the case of an elliptic equation the conditions of existence and uniqueness of the solution for the problem (1)-(6) are established.

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Multidimensional analogue of Thron's theorem about twin parabolic convergence regions for continued fractions

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Branched continued fractions (BCF) is a multidimensional generalisation of continued fractions. Lately, it is intensively developed the theory of BCF of the special form with complex elements

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}},$$

where $i_0 = N$ is a fixed natural number.

Theorem. *Let the elements of the BCF*

$$\left(b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{1}\right)^{-1} \quad (1)$$

lie in parabolic domains, that is $a_{i(k)} \in \mathcal{P}_{i(k)}$, $i(k) \in \mathcal{I}$, *where*

$$\mathcal{P}_{i(k)} = \left\{ z \in \mathbb{C} : |z| - \mathrm{Re}\left(ze^{-2i\gamma}\right) < \frac{2D_k^2}{i_{k-1}}(1-\varepsilon)\cos^2\gamma \right\},$$

$$\mathcal{I} = \{i(k) :\ i(k) = (i_1, i_2, \ldots, i_k),\ 1 \le i_p \le i_{p-1},\ 1 \le p \le k,\ i_0 = N\},$$

where $D_{2s}=(1-d)^2,\ D_{2s+1}=d^2,\ 0<d<1,\ s=1,2,\ldots$; *the* ε *is an arbitrary small real number* $(0<\varepsilon<1)$. *Then*

1) there exist finite limits of even and odd approximants of the BCF (1)*;*

2) the BCF (1) *converges if the following series diverge*

$$\sum_{p=1}^{\infty}\left|a_{m[p+1]}\right|^{-1/2},\ m=\overline{1,N},$$

$$\sum_{p=1}^{\infty}\left|a_{i(n),m[p+1]}\right|^{-1/2},\ i(n)\in\mathcal{I}^{(m+1)},\ m=\overline{1,N-1};$$

$$\mathcal{I}^{(m+1)}=\{i(k)=(i_1,i_2,\ldots,i_k)\ :m+1\le i_k\le i_{k-1}\le\ldots\le i_0;\ k\ge 1;\ i_0=N\},$$

$m[p]=\underbrace{m,m,...,m}_{p};\ p=1,2,\ldots$.

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Generalized Hermite polynomials in the description of fractional calculus and other applications

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Using the concepts and formalism of different families of Hermite polynomials, we here discuss how to represent the action of the operators involving fractional derivatives and we introduce a family of polynomials strictly related to the Hermite polynomials in order to compute the effect of fractional operators on a given function. Finally, we present some generalizations of polynomials belonging to the Bernoulli class. In particular, by using the generating function method and the concept and formalism of the two-variable Hermite polynomials, we introduce the generalized Bernoulli polynomials.

Two-dimensional generalized moment representations and Pad'e approximations for pseudo-twovariate functions

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Generalized moment representations are used to study the so-called pseudo-twovariate functions

$$f(z,w)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\widetilde{s}_{k+m}z^k w^m=\frac{z\widetilde{f}(z)-w\widetilde{f}(w)}{z-w},$$

where

$$\widetilde{f}(z)=\sum_{k=0}^{\infty}\widetilde{s}_k z^k.$$

We construct Padé-type approximants explicitly for Humbert confluent hypergeometric series [1]

$$f(z,w) = \Phi_2(1,1,\nu+\sigma+2,z,w) = \frac{z\ {}_1F_1(1;\nu+\sigma+2;z) - w\ {}_1F_1(1;\nu+\sigma+2;w)}{z-w}, \quad \nu,\sigma > -1.$$

We consider the partial case for $\nu + \sigma = -1$:

$$f(z,w) = \frac{we^w - ze^z}{w-z}. \tag{1}$$

Note that the numerical examples presented in [2] are connected with the computation of rational approximations obtained as certain two-dimensional generalizations of Pad'e approximations for the function (1).

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Adaptive approximation by sums of piecewise polynomials on sparse grids

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Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, be a bounded domain. We call a partition $\triangle$ of Ω *convex* if every cell $\omega \in \triangle$ is convex. For $N \in \mathbb{N}$, denote by $\mathcal{P}_N$ the set of all convex partitions of Ω comprising at most N cells.

For a function $f \in L_p(\Omega)$ we define its *N-term approximation error* by

$$\sigma_N(f,\mathcal{D})_p = \inf_{g_i \in \mathcal{D}, i=1,\ldots,N} \Big\| f - \sum_{i=1}^{N} c_i g_i \Big\|_{L_p(\Omega)},$$

where $c_i \in \mathbb{R}$ for $i = 1,\ldots,N$ and $\mathcal{D}$ is an arbitrary set of functions in $L_p(\Omega)$ (*dictionary*). In our case of piecewise constant approximation, we consider dictionaries: $\mathcal{D}_C$ – set of characteristic functions of arbitrary convex sets, and $\mathcal{D}_S$ – set of characteristic functions of arbitrary simplexes.

It is easy to see that

$$\sigma_N(f,\mathcal{D}_C)_\infty \geqslant \frac{const}{N}$$

for any measurable function.

It has been shown in [1] that piecewise constants on a partition which consist of N convex polyhedra provide the L_p-approximation order $O(N^{-2/(d+1)})$ for functions from Sobolev space W_q^2, where d is the number of variables, $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$ satisfy inequality

$$\frac{2}{d+1} + \frac{1}{p} - \frac{1}{q} \geqslant 0.$$

This order cannot be further improved for any function whose Hessian is positive definite at some point [2]. This implies $\sigma_N(f,\mathcal{D}_S) = O(N^{2/(d+1)})$ for such functions. Although still suffering from the curse of dimensionality, this bound is significantly better than the standard order $O(N^{-1/d})$ expected from piecewise constants on isotropic partitions.

On the other hand, it is easy to see that piecewise constant sparse grids approximation implies $\sigma_N(\mathcal{D}_S)_p = O(N^{-1}\ln^{2(d-1)} N)$ for functions in Sobolev spaces with dominating mixed derivatives. We improve this bound and show that piecewise constant sparse grids approximation as linear combinations of Haar tensor product functions leads to to

$$\sigma_N(\mathcal{D}_S)_p = O(N^{-1}\ln^{3(d-1)/2} N)$$

for $1 < p < \infty$. Case $d = 2$, $p = 2$ was previously proved in [3]. Also, using a modification of the sparse grid approximation by employing some techniques of [1,2], in the 2D case we improve this error from

$$\sigma_N(f, \mathcal{D}_S)_\infty = O(N^{-1}\ln^2 N) \quad \text{to} \quad \sigma_N(f, \mathcal{D}_S)_\infty = O(N^{-1}\ln N).$$

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Interpolation rational integral fraction of Hermitian-type on a continuum set of nodes

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The number of publications are devoted to the approximation of functionals

$$F : L_1(0,1) \to \mathbb{R}^1$$

on a continuum set of nodes

$$x^n(z, \boldsymbol{\xi}^n) = x_0(z) + \sum_{i=1}^{n} H(z - \xi_i)\big[x_i(z) - x_{i-1}(z)\big],$$
$$\boldsymbol{\xi}^n = (\xi_1, \xi_2, \ldots, \xi_n) \in \boldsymbol{\Omega}_{\boldsymbol{z}^n} = \{\boldsymbol{z}^n : 0 \le z_1 \le \ldots \le z_n \le 1\},$$

for example [1]–[3].

In our research

$$x_i(z) \in Q[0,1], \quad i = 0, 1, \ldots,$$

are arbitrary fixed elements of the space $Q[0,1]$ of piecewise continuous functions on a segment $[0,1]$ with a finite number of discontinuity points of the first kind. The set of such functions is called the interpolant framework and $H(t)$ is a Heaviside function.

These works can be divided into two groups. The first group includes papers in which polynomial interpolation of functionals is investigated. The second group includes the works that are devoted to representation of functionals by chain fractions.

The purpose of this research is to construct and study an integral rational approximation of a functional on a continuum set of nodes, which is the ratio of a functional polynomial of the first degree to a functional polynomial of the second degree.

The kernels under the integral are determined from the corresponding continuum conditions. In this case, we obtain an integral equation to determine the kernel of the numerator integral. This integral equation using simple transformations is reduced to the standard form of the integral Volterra equation of the second kind. The lemma on the existence of a single solution of this equation is proved.

Substituting the obtained solution into expressions for the rest of the kernels, we obtain expressions for all kernels included in the integral rational interpolant. In this case, in order for a rational third-order functional to be interpolation on continuum nodes, it is sufficient for this functional to satisfy the substitution rule.

Note that the resulting interpolant is such that holds any rational functional of the obtained form.

To obtain a functional interpolation rational interpolant with two double interpolation nodes, we use continuous Hermitian-type interpolation conditions. Then, in order for a rational third-order functional to be interpolation on continuous nodes, it is sufficient for this functional to satisfy the substitution rule.

Note again that the resulting interpolant is such that holds any rational functional of the obtained form.

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Best approximations by analytic vectors relative to operators with discrete spectrum

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In a Banach space $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ we consider a closed linear operator $A\colon \mathfrak{D}^1(A) \to \mathfrak{X}$ with dense domain $\mathfrak{D}^1(A)$. We assume that A has a discrete spectrum $\sigma(A)$, i.e., its resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ has only isolated eigenvalues $\{\lambda_j \in \mathbb{C}\colon j \in \mathbb{N}\}$ of finite multiplicities, which are poles with the limit at infinity.

For any $\nu > 0$ and $k \in \mathbb{Z}_+$ we put

$$x_{k,\nu} := (A/\nu)^k x, \quad x \in \mathfrak{D}^\infty(A) := \bigcap_{k\in\mathbb{Z}_+} \mathfrak{D}^k(A).$$

Let $\{x^*_{k,\nu}\}_{k\in\mathbb{Z}_+}$ denotes the rearrangement of the elements by magnitude of the norms:

$$\|x^*_{0,\nu}\|_{\mathfrak{X}} \geq \|x^*_{1,\nu}\|_{\mathfrak{X}} \geq \ldots \geq \|x^*_{k,\nu}\|_{\mathfrak{X}} \geq \ldots .$$

For $1 < q < \infty$ and $1 \leq p \leq \infty$ the subspaces $\mathcal{E}^\nu_{q,p}(A)$ of analytic vectors relative to A have the following form:

$$\mathcal{E}^\nu_{q,p}(A) = \left\{ x \in \mathfrak{X}\colon \|x\|_{\mathcal{E}^\nu_{q,p}(A)} = \left(\sum\nolimits_{k\in\mathbb{N}} \|x^*_{k-1,\nu}\|^p_{\mathfrak{X}} k^{\frac{p}{q}-1}\right)^{1/p} < \infty \right\}$$

(specified also for $p = \infty$).

We prove the inequalities that provide an accurate estimate of errors for the best approximations by the subspaces $\mathcal{E}^\nu_{q,p}(A)$. The obtained estimates of approximation errors are expressed in terms of the quasinorms of the approximation spaces $\mathcal{B}^s_{q,p,\tau}(A)$ associated with A (see [1]).

The essential in our approach is that the approximation spaces $\mathcal{B}^s_{q,p,\tau}(A)$ can be identified with the interpolation spaces obtained by the real method of interpolation.

For $0 < \theta < 1$, we use the normalisation factor

$$N_{\theta,r} := \begin{cases} \left(\int_0^\infty t^{r(1-\theta)-1}(1+t^2)^{-r/2} dt \right)^{-1/r}, & 1 \le r < \infty, \\ \theta^{-\theta/2}(1-\theta)^{-(1-\theta)/2}, & r = \infty, \end{cases}$$

to write the constants in estimates of approximation errors. This normalisation factor is determined by the parameters τ and s of the approximation spaces

$$\mathcal{B}^s_{q,p,\tau}(A) \quad (\theta = 1/(1+s), r = \tau/\theta)$$

and has a form

$$N_{\theta,2} = ((2/\pi)\sin(\pi\theta))^{1/2}$$

in the case $r = 2$.

Exact estimates for approximation errors of spectral approximations for unbounded operators in Banach spaces, using the Besov-type quasinorms and normalisation factor

$$N'_{\theta,r} = [r\theta(1-\theta)]^{1/r}$$

for $1 \le r < \infty$ and $N'_{\theta,\infty} = 1$, are given in [1]. The normalisation factor $N_{\theta,r}$ is used in [2] to study the approximation problem by invariant subspaces of analytic vectors relative to positive operators in Banach spaces.

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Approximation of analytic functions of several variables by multidimensional regular C-fractions with independent variables

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Let N be a fixed natural number. Let us introduce the following sets of multiindices:

$$\mathcal{I}_k = \{i(k) : \ i(k) = (i_1, i_2, \ldots, i_k), \ 1 \le i_p \le i_{p-1}, \ 1 \le p \le k, \ i_0 = N\}, \quad k \ge 1.$$

A branched continued fraction of the form

$$\sum_{i_1=1}^{N} \frac{a_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{a_{i(3)} z_{i_3}}{1} + \cdots,$$

where the $a_{i(k)} \in \mathbb{C} \setminus \{0\}$, $i(k) \in \mathcal{I}_k$, $k \ge 1$, and $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, is called a multidimensional regular C-fraction with independent variables.

In [1, 2, 3] we obtained some estimates of the rate of convergence for multidimensional regular C-fractions with independent variables in some regions (here region is a domain together with all, part or none of its boundary). We also constructed an algorithm for the expansion of the given formal multiple power series into the so-called corresponding multidimensional regular C-fraction with independent variables (see [4]). A few numerical experiments show, on the one hand, the efficiency of the proposed algorithm and, on the other, the power and feasibility of the method in order to numerically approximate certain analytic functions of several variables from their formal multiple power series.

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Simultaneous approximation by operators

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In 1912 S. Bernstein constructed a sequence of algebraic polynomials associated to a function to give a very simple proof of the fundamental Weierstrass approximation theorem about the uniform approximation of continuous functions on compact intervals by algebraic polynomials. For $f \in C[0,1]$ and $n \in \mathbb{N}$ these polynomials, now known as the Bernstein polynomials, are defined by

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}, \quad x \in [0,1].$$

We have

$$\lim_{n\to\infty} B_n(f,x) = f(x) \quad \text{uniformly on } [0,1].$$

Later on, it was observed that the derivatives of the Bernstein polynomials similarly approximate the corresponding derivatives of the function, provided that it is smooth enough. This phenomenon is known as simultaneous approximation. I will present results that describe the rate of this approximation. For example, in the uniform norm $\|\circ\|$ on $[0,1]$, they have the form

$$\|(B_nf)' - f'\| \le c\left(\omega_\varphi^2(f', n^{-1/2}) + \omega_1(f', n^{-1})\right)$$

and

$$\|(B_nf)^{(s)} - f^{(s)}\| \le c\left(\omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega_1(f^{(s)}, n^{-1}) + \frac{1}{n}\|f^{(s)}\|\right), \quad s \ge 2,$$

assuming that f has continuous derivatives up to the indicated order. Moreover, these estimates cannot be improved. Above, $\omega_1(F,t)$ is the classical modulus of continuity of F in the uniform norm on $[0,1]$, and $\omega_\varphi^2(F,t)$ is the Ditzian-Totik modulus of smoothness of second order with step-weight $\varphi(x) := \sqrt{x(1-x)}$ in the same norm. Actually, such results are valid in the L_p-norm with Jacobi-weights under certain natural restrictions on the weight exponents.

I will also present similar relations about the Baskakov and the Szász-Mikarjan operators.

The last topic I will address is the fascinating subject of the approximation of functions by algebraic polynomials with *integer* coefficients. In 1931 L. Kantorovich introduced the following modification of the Bernstein polynomials

$$\widetilde{B}_n(f,x) := \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right)\binom{n}{k}\right] x^k(1-x)^{n-k}, \quad x \in [0,1],$$

where $[\alpha]$ denotes the largest integer that is less than or equal to the real α. L. Kantorovich showed that $\widetilde{B}_n(f)$ approximates f in the uniform norm on $[0,1]$ iff $f(0)$ and $f(1)$ are integers. I will present an extension of this result with regard to simultaneous approximation.

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Approximative characteristics of the Nikol'skii-Besov-type classes of periodic functions

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We study the classes $B^{\Omega}_{p,\theta}$ of periodic functions of several variables [1] with

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j\right),$$

where ω is a given function (of one variable) of the type of a modulus of continuity of order l that satisfies the conditions (S^{α}) and (S_l), which are called the Bari-Stechkin conditions [2]. For a certain choice of function Ω, the classes $B^{\Omega}_{p,\theta}$ coincide with analogs of the well-known Nikol'skii-Besov classes $B^{r}_{p,\theta}$ [3].

Let

$$L_{\infty}(\pi_d), \quad \pi_d = \prod_{j=1}^{d}[0; 2\pi),$$

be the space of essentially bounded functions $f(x) = f(x_1, ..., x_d)$, which are 2π-periodic in each variable, with the norm

$$\|f\|_{\infty} = \operatorname*{ess\,sup}_{x\in\pi_d} |f(x)|.$$

Let $\{u_i\}_{i=1}^{M}$ be an orthonormal system of functions $u_i \in L_{\infty}(\pi_d)$, $i = \overline{1, M}$, and let

$$\sum_{i=1}^{M}(f, u_i)u_i$$

be the orthogonal projection of the function f onto the subspace generated by the system of functions $\{u_i\}_{i=1}^{M}$.

We obtain exact order estimates of orthowidths of the classes $B^{\Omega}_{p,\theta}$ in the space $B_{\infty,1}$, which norm is stronger than the L_{∞}-norm. For the functional classes $B^{\Omega}_{p,\theta} \subset B_{\infty,1}$, these quantities are defined as follows:

$$d^{\perp}_{M}(B^{\Omega}_{p,\theta}, B_{\infty,1}) = \inf_{\{u_i\}_{i=1}^{M}} \sup_{f\in B^{\Omega}_{p,\theta}} \left\| f(\cdot) - \sum_{i=1}^{M}(f, u_i)u_i(\cdot)\right\|_{B_{\infty,1}}.$$

The following theorem holds true.

Theorem. *Let*

$$d \geq 1, \quad 1 < p < \infty, \quad 1 \leq \theta \leq \infty, \quad \Omega(t) = \omega\left(\prod_{j=1}^{d} t_j\right),$$

where ω satisfies condition (S^{α}) with some $\alpha > 1/p$ and condition (S_l). Then for any $M, n \in \mathbb{N}$, such that $M \asymp 2^n n^{d-1}$, the relation

$$d^{\perp}_{M}(B^{\Omega}_{p,\theta}, B_{\infty,1}) \asymp \omega(2^{-n})2^{n/p}n^{(d-1)(1-1/\theta)}$$

holds.

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Approximation by discrete operators of Max-Min kind

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In this presentation, we aim to obtain the approximation to the multivariate functions via max-min type discrete operators and for this, we also investigate the proper conditions provided by the discrete kernels. Additionally, we estimate the rate of convergence. We also note that the papers [1–3] are the main motivation of this research and some approximation studies with max-min operators satisfying the pseudo-linearity are given in [3, 4]. At the end of this study, using these operators, we also give graphs for the approximation to univariate and bivariate functions with discrete kernels that satisfy the conditions.

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Some Old and New Methods Concerning Convergence of Linear Positive Operators

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The theory of approximation is closely related to many fields. After the introduction of Bernstein polynomials many other exponential type operators were introduced and their approximation behaviour have been discussed. As such the exponential type operators also include discrete type operators, which were appropriately modified to approximate integrable functions. We discuss here some old and new problems concerning convergence of several linear positive operators. We also mention the applications of quantum and post-quantum calculus, in the approximation theory. Finally we discuss about the difference estimates between the two operators.

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Isomorphisms of some algebras of analytic functions of bounded type on Banach spaces

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Let $\mathbb{I} = \{I_1, I_2, \ldots, I_n, \ldots\}$ be a sequence of algebraically independent polynomials on a Banach space Z, such that $\|I_n\|_1 = 1$, $n \in \mathbb{N}$. We denote by $\mathcal{I}_{\mathbb{I}}(Z)$ the minimal unital algebra containing polynomials in $\mathbb{I}$. Let $H_{b\mathbb{I}}(Z)$ be the closure of $\mathcal{I}_{\mathbb{I}}(Z)$ in $H_b(Z)$.

We consider conditions under which two algebras $H_{b\mathbb{A}}(X)$ and $H_{b\mathbb{P}}(Y)$ are isomorphic via a mapping Θ such that $\Theta(A_n) = P_n$, $n \in \mathbb{N}$. Some applications for algebras of symmetric analytic functions of bounded type are obtained.

On convergence of an expansion of function $F_4(1,2;2,2;z_1,z_2)$ into a branched continued fraction

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An expansion of the hypergeometric Appell function $F_4(1,2;2,2;z_1,z_2)$ [1] into a branched continued fraction with two branches on an even level and one branch on an odd level is constructed. The formulas of fraction coefficients are obtained.

$$\cfrac{1}{1-\cfrac{z_1}{1-\cfrac{1/2z_2}{1-\ddots}}-\cfrac{z_2}{1-\cfrac{z_1}{1-\ddots}}}.$$

The theorem on the convergence of the obtained branched continued fraction in the Vorpitski circle [2] is proved. A numerical analysis was performed. The values of the approximants branched continued fractions and the values of the corresponding partial sums of the hypergeometric series [1] at different points of the two-dimensional complex space were calculated. The calculations were performed both inside and outside the established convergence region. According to the results of calculations, we can assume the existence of a wider range of convergence of the obtained fraction.

№	(z_1, z_2)	$F_4(1,2;2,2;z_1,z_2)$	f_{n_1}	n_1	S_{n_2}	n_2
1	(-0.1867-0,1867*i, -0.1867+0*i)	0.732736786- -0.089504072*i	0.732736786- -0.089504072*i	8	0.732736458- -0.089503907*i	57
2	(0.125+0,125*i, 0.125+0,125*i)	1.148753955+ 0,43019177*i	1.148753955+ 0,43019177*i	10	1.148755406+ 0,43019259*i	28
3	(-2.125-0,125*i, -0.125-0,125*i)	0.314931823- 0,016463586*i	0.314931823- 0,016463586*i	10	not exists	-
4	(-2.2596+0,9712*i, -0.7212+0,8173*i)	0.207231107- 0,122449614*i	0.207231107- 0,122449614*i	16	not exists	-
5	(0.25+0.25*i, 0.05+0.05*i)	1.15609905+ 0.53305734*i	1.15609905+ 0.53305734*i	9	1.15609920+ 0.53305637*i	32

Figure 1: Results of calculate the values of function $F_4(1,2;2,2;z_1,z_2)$

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Approximation of functions from the Lipschitz classes by their conjugate Poisson integrals

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The talk is devoted to the results concerning one of the important problems of the approximation theory, namely, finding asymptotic expansions for the value of best approximation of functions from some classes using linear methods of summation of the Fourier series.

We consider the functions belonging to the well-known Lipschitz class $\mathrm{Lip}_1\alpha$, $0 < \alpha \le 1$, i.e. continuous 2π–periodic functions f, such that

$$\forall\, t_1, t_2 \in \mathbb{R} \qquad |f(t_1) - f(t_2)| \le |t_1 - t_2|^{\alpha}$$

holds. Then, we approximate the functions by their conjugate Poisson integrals

$$\overline{P}_{\rho}(f;x) = P_{\rho}(\overline{f};x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\overline{K}_{\rho}(t)\,dt,$$

where

$$\overline{K}_{\rho}(t) = \sum_{k=1}^{\infty}\rho^k \sin kt = \frac{\rho\sin t}{1-2\rho\cos t+\rho^2}, \qquad 0 \le \rho < 1,$$

is the kernel.

In [1], the complete asymptotic expansions in terms of $\ln\frac{1}{\rho}$ as $\rho \to 1-$ of the quantity

$$\mathcal{E}(\mathrm{Lip}_1\,\alpha; \overline{P}_{\rho})_C = \sup_{f\in \mathrm{Lip}_1\,\alpha} \left\|\bar{f}(\cdot) - \overline{P}_{\rho}(f;\cdot)\right\|_C, \qquad 0 < \alpha \le 1,$$

are obtained, where

$$\bar{f}(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cot\frac{t}{2}\,dt$$

is the conjugate to f function, and the approximation error is measured in the uniform norm

$$\|f\|_C = \max_t |f(t)|.$$

As a consequence, we get estimates of the quantity $\mathcal{E}(\mathrm{Lip}_1 1; \overline{P}_\rho)_C$ in a partial case, for $\alpha = 1$, and write down the expansion in terms of the generalized Riemann zeta function.

These expansions make it possible to write down the Kolmogorov–Nikol'skii constants of the arbitrary order of smallness.

Note, that this research complement the corresponding results from [2] for the classes $\mathrm{Lip}_1 1$, where the asymptotic expansion in terms of $(1-\rho)$ as $\rho \to 1-$ of approximation of functions by conjugate Poisson integrals was obtained.

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Urysohn type operators and their asymptotic properties

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In the present work, it is aimed to examine the convergence properties of the Urysohn type integral operators, to which the Ukrainian mathematicians contributed greatly to the definition and development of the theory of Urysohn integral operators. As a continuation of recent studies of the author, we mainly focus our attention on the investigation of some Urysohn type operators and their asymptotic properties. The basis arguments used in these investigations are the Fréchet and Prenter Density theorems, Urysohn type operator values and some methods from approximation theory developed especially by Italian mathematicians in recent years.

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Brached continued fractions and Z index of the tree graph

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The concept of the topological index was first introduced by Haruo Hosoya in 1971 [2]. Later more different types of indices have been discovered in chemical graph theory. The first topological index is also called Hosoya index or the Z index nowadays, which is defined by

$$Z = \sum_{k=0}^{m} p(G, k),$$

where $p(G, k)$ is the number of ways of choosing k disjoin edges from a graph G. Hosoya [3] also showed that Z index can be calculated for some tree graphs including path graph, cycle graph and caterpillar graph G by using the continued fraction.

In this talk, we show how to calculate more complicated graphs by using more general continued fractions ([1, 5, 6, 7]). In fact, if there is one to one correspondence between a given graph and a branched or two-dimensional continued fraction, its Z index can be calculated systematically and efficiently.

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Ostrowski type inequalities for sets and functions of bounded variation

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We introduce a new definition of variation for multidimensional sets and functions, which is based on the Kronrod-Vitushkin approach. The introduced variation of a multivariate function has (unlike any of the Kronrod-Vitushkin variations) the following two properties: the variation does not change if the argument of the function is multiplied by a non-zero constant; and the variation of a multivariate radial function is twice bigger than the variation of the generating one-dimensional function.

For natural $d \geq 2$ denote by B^d the unit ball in $\mathbb{R}^d$; for $p \in [1, \infty]$ by $v_p(F)$ and $v_p(f)$ we denote the introduced variation of a compact set $F \subset B^d$ and of a continuous function $f\colon B^d \to \mathbb{R}$. In [1] the following sharp following Ostrowski type inequalities were proved.

Theorem 1. *Let $d \in \mathbb{N}$ and a closed set $F \subset B^d$ be given. If $\theta \notin F$, then for all $p \in [1, \infty]$*

$$\mu^d F \leq \frac{\mu^d B^d}{2} v_p(F).$$

The inequality is sharp. If equality holds, then $\mu^d F = 0$.

Theorem 2. *Let $d \in \mathbb{N}$ and a continuous function $f\colon B^d \to \mathbb{R}$ be given. Then for all $p \in [1, \infty]$*

$$\left| \frac{1}{\mu^d B^d} \int\limits_{B^d} f(x)dx - f(\theta) \right| \leq \frac{v_p(f)}{2}.$$

The inequality is sharp. It becomes equality only in the case when f is constant.

In [2] these results were generalized to a class of domains that are more general than B^d.

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Three-dimensional regular C-fractions

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Some number of analytic functions of several variables are known to have many-dimensional continued fraction representations. Frequently a given function will be represented by several different many-dimensional continued fractions, each with its own behavior. One of the types of functional three-dimensional continued fractions correspondent to a formal triple power series is considered

$$\frac{a_{0,0,0}}{F_0(z_1, z_2, z_3) + \underset{i=1}{\overset{\infty}{\mathrm{D}}} \dfrac{a_{i,i,i} z_1 z_2 z_3}{F_i(z_1, z_2, z_3)}}, \tag{1}$$

$$F_i(z_1, z_2, z_3) = 1 + \underset{j=1}{\overset{\infty}{\mathrm{D}}} \frac{a_{i+j,i,i} z_1}{1} + \underset{j=1}{\overset{\infty}{\mathrm{D}}} \frac{a_{i,i+j,i} z_2}{1} + \underset{j=1}{\overset{\infty}{\mathrm{D}}} \frac{a_{i,i,i+j} z_3}{1}$$

$$+ \underset{j=1}{\overset{\infty}{\mathrm{D}}} \frac{a_{i+j,i+j,i} z_1 z_2}{\Phi_{i+1,i+1,i}(z_1, z_2)} + \underset{j=1}{\overset{\infty}{\mathrm{D}}} \frac{a_{i+j,i,i+j} z_1 z_3}{\Phi_{i+1,i,i+1}(z_1, z_3)} + \underset{j=1}{\overset{\infty}{\mathrm{D}}} \frac{a_{i,i+j,i+j} z_2 z_3}{\Phi_{i,i+1,i+1}(z_2, z_3)},$$

$$\Phi_{k,k,i}(z_1,z_2) = 1 + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{k+j,k,i} z_1}{1} + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{k,k+j,i} z_2}{1},$$
$$\Phi_{k,i,k}(z_1,z_3) = 1 + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{k+j,i,k} z_1}{1} + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{k,i,k+j} z_3}{1},$$
$$\Phi_{i,k,k}(z_2,z_3) = 1 + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{i,k+j,k} z_2}{1} + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{i,k,k+j} z_3}{1},$$

where $a_{i,j,k} \neq 0$, $i,j,k = 0,1,\ldots$, $z = (z_1,z_2,z_3) \in \mathbb{C}^3$.

The fraction (1) with complex coefficients $a_{i,j,k}$ is called three-dimensional regular C-fraction (ThDR C-F).

If all coefficients $a_{i,j,k} \geq 0$ and $z = (z_1,z_2,z_3) \in \mathbb{C}^3$ the fraction (1) is called three-dimensional S-fraction.

Using bounds for the three-dimensional regular C-fraction (1) tails, a difference formula for two approximants of the ThDR C-F in terms of its tails, and the principle of correspondence, properties of such fractions are investigated.

Theorem 1. *Every ThDR C-F (1) corresponds to a uniquely determined formal triple power series (ftps):*

$$P(z) = \sum_{|k|=0}^{\infty} c_k z^k, \tag{2}$$

where

$$z = (z_1,z_2,z_3) \in \mathbb{C}^3, \quad k = (k_1,k_2,k_3) \in \mathbb{Z}_+^3, \quad |k| = k_1+k_2+k_3, \quad z^k = z_1^{k_1} z_2^{k_2} z_3^{k_3}.$$

The order of correspondence of the nth approximant of (1) $f_n(z)$ is

$$\nu_n = n+1.$$

Theorem 2. *Let ThDR C-F (1) converges uniformly on some bounded domain $D \in \mathbb{C}^3$ to a holomorphic function $f(z)$, $z = (z_1,z_2,z_3) \in D$. Then corresponding to (1) ftps (2) also converges to the function $f(z)$ on D.*

Approximation properties of multivariate sampling type operators

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In this paper, we introduce multivariate form of generalized sampling type operators and study pointwise and uniform convergence as well as rate of convergence for the functions belong to space of log-uniformly continuous functions. Furthermore, we state and prove the generalized Mellin Taylor's expansion and using this expansion we establish pointwise asymptotic behaviour of the series by means of Vonovskaya type theorem.

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Convergence of the continued fraction for the ratio of confluent hypergeometric functions over the field of P-adic numbers

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We consider the following continued fraction [1]

$$b_0(z) + \underset{n=1}{\overset{\infty}{\mathrm{D}}} \frac{a_n(z)}{b_n(z)}, \quad z \in \mathbb{C}, \tag{1}$$

where $a_0(z) \equiv 1$, $b_0(z) \equiv 0$,

$$a_{2k}(z) = (a - b - k + 1)z, \quad a_{2k+1}(z) = (a + k)z, \quad b_k(z) = b + k, \quad k \geq 1, \tag{2}$$

and a, b are complex numbers, such that $b \notin -\mathbb{N}$.

The fraction (1) arises from the expansion of the ratio of confluent hypergeometric functions

$$\frac{{}_1F_1(a+1, b+1; z)}{b\, {}_1F_1(a, b; z)} \tag{3}$$

into continued fraction [1]. We will remind that confluent hypergeometric function ${}_1F_1(a, b; z)$ is an entire function of a or z, except when $b = 0, -1, -2, \ldots$, and given by the series

$${}_1F_1(a, b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \tag{4}$$

where $(\cdot)_n$ are Pochhammer symbols.

The aim of our work is to establish the conditions of convergence of the fraction (1) in the case when the parameters a, b and variable z are p-adic numbers [2] and the convergence of sequence of approximants is considered in the p-adic norm. The field of p-adic numbers, denoted by the symbol $\mathbb{Q}_p$, is defined as the completion of the field $\mathbb{Q}$ with respect to the p-adic norm [2] and is non-archimedean.

The case of continued fraction for the ratio of Gauss hypergeometric functions ${}_2F_1(a, b, c; z)$ over the $\mathbb{Q}_p$ was investigated in [3].

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Best approximations on abstract Wiener spaces

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We establish inverse and direct theorems in the form of Bernstein-Jackson inequalities with exact constants for estimations of best approximations in compatible interpolation couples (A_0, A_1) of quasinormed Abelian groups. For this purpose, we consider the scale of Besov type approximation Abelian groups

$$\mathcal{B}^s_\tau(A_0,A_1) = (A_0,A_1)^{1/\vartheta}_{\vartheta,q}, \quad \vartheta = 1/(s+1),\ \tau = \vartheta q,\ 0<\vartheta\le 1,\ 0<q\le\infty$$

endowed with the quasinorm $\|\cdot\|_{\mathcal{B}^s_\tau}$ defined by the real interpolation method, using the E-functional

$$E(t,a;A_0,A_1) := \inf\left\{\|a-a_0\|_{A_1} : a\in A_1,\ \|a_0\|_{A_0}<t\right\}.$$

Then the Bernstein-Jackson-type inequalities obtain the form

$$\|a\|_{\mathcal{B}^s_\tau} \le c_{s,\tau}\|a\|^s_{A_0}\|a\|_{A_1} \quad \text{for all} \quad a\in A_0\cap A_1,$$
$$E(t,a;A_0,A_1) \le t^{-s}2^{s+1}C_{s,\tau}\|a\|_{\mathcal{B}^s_\tau} \quad \text{for all} \quad a\in \mathcal{B}^s_\tau(A_0,A_1)$$

with asymptotically sharp at $\tau\to\infty$ the constants $c_{s,\tau}, C_{s,\tau}$ calculated in [1].

These results are applied to approximations in quasi-Banach spaces $L_{p_1}(\gamma)$ with $0<p_1\le\infty$ of functions in Gaussian random variables $X\ni x\mapsto\phi_h(x)$ $(h\in H)$ on an abstract Wiener space (X,H,γ) with a reproducing kernel Hilbert space H, that are characterized by the property $\int_X \exp(\mathbb{i}\phi_h)\,d\gamma = \exp\left(-\|h\|^2/2\right)$. The Wiener space (X,H,γ) is endowed with the Gaussian measure γ on a separable real Banach space X, according to Gross's theory [2].

In this case, we take instead (A_0,A_1) the compatible couple of quasi-Banach spaces $(\mathscr{E}_{p_0}(\nabla_\Lambda), L_{p_1}(\gamma))$ with different $p_0\in(1,\infty)$, $p_1\in(0,\infty]$ in which

$$\mathscr{E}_{p_0}(\nabla_\Lambda)\subset L_{p_0}(\gamma;\Gamma), \quad \mathscr{E}_{p_0}(\nabla_\Lambda) := \bigcup\left\{\mathscr{E}^\nu_{p_0}(\nabla_\Lambda)\colon \nu>0\right\}$$

consist of analytic in Nelson's sence functions of exponential type in relative to the Malliavin gradient ∇_Λ modified by a linear operator $\Lambda\colon \mathscr{D}(\Lambda)\subset H\to H$ [3], where

$$\mathscr{E}^\nu_{p_0}(\nabla_\Lambda) := \left\{f\colon \limsup_{r\to\infty}\frac{1}{r}\ln\left(\max_{|t|=r}|\tilde f(t)|\right)\le\nu\right\}, \quad \tilde f(t) := \sum_{k\ge 0}\frac{t^k}{k!}\|\nabla^k_\Lambda f\|_{L_{p_0}(\gamma;\Gamma)}$$

and ∇_Λ is a closable unbounded operator acting from $L_{p_0}(\gamma)$ to the space $L_{p_0}(\gamma;\Gamma)$ of functions with values in the symmetric Fock space $\Gamma(H)$ generated by H (here, $\mathscr{E}_{p_0}(\nabla_\Lambda)$ contains the total family of Hermite polynomials on X which ensures its non-triviality). In this case the interpolation structure of the suitable Besov space

$$\mathcal{B}^s_\tau(\gamma) = (\mathscr{E}_{p_0}(\nabla_\Lambda), L_{p_1}(\gamma))^{1/\vartheta}_{\vartheta,q}, \quad \vartheta = 1/(s+1),\ \tau=\vartheta q,\ 0<\vartheta\le 1,\ 0<q\le\infty$$

allows to establish the exact analogue of Bernstein-Jackson-type inequalities that fully characterize rapidity of approximations in the quasi-Banach spaces $L_{p_1}(\gamma)$ over abstract Wiener spaces (X,H,γ) by functions of exponential type from $\mathscr{E}_{p_0}(\nabla_\Lambda)$.

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Green's matrix for systems of Kolmogorov-type equations

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We investigate the structure of the fundamental matrices of the solutions of the Cauchy problem for degenerate parabolic equations of the diffusion type with inertia.

Let

$$n_j \in N, \quad n \in N, \quad n_1 \geq ... \geq n_p, p \in N, \quad j = \overline{1,p},$$
$$x = (x_1, ..., x_p), \quad x_j = (x_{11}, ..., x_{nj}) \in R^{n_j}, \quad \xi = (\xi_1, ..., \xi_p), \quad \xi_j \in R^{n_j},$$
$$n_0 = \sum_{j=1}^{p} n_j.$$

Consider the Cauchy problem

$$Lu = \partial_t u\,(t,x) - \sum_{j=1}^{p-1} \sum_{l=1}^{n_{j+1}} x_{jl} \partial_{x_{j+1}l} u\,(t,x) - \sum_{|k| \leq 2b} A_k\,(t,x) D_{x_1}^k u\,(t,x) = 0, \tag{1}$$

$$u(t,x)|_{t=\tau} = u_0(x), \quad 0 \leq \tau \leq t \leq T < +\infty, x \in R^{n_0}, \tag{2}$$

where the system

$$\partial_t u(t,x) = \sum_{|k| \leq 2b} A_k(t,x) D_{x_1}^k u(t,x), \tag{3}$$

is uniformly parabolic in the sense of Petrovsky in the strip

$$\Pi_{[0,T]} = \{(t,x), x \in R^{n_0}, 0 \leq t \leq T\},$$

$(x_2, ..., x_p)$-parameters.

Let

$$\rho_{1j}(t,\sigma,c) = \sum_{\mu=1}^{k-2} \frac{t^{\mu-1}}{(\mu-1)!} C_{1j\mu} + \frac{t^{k-1}}{(k-1)!} \sigma_{kj}, j = \overline{n_{k+1}, n_k}, k = \overline{2,p}, \quad \rho_{ij} = \sigma_{1j}, n_2 \leq j \leq n_1.$$

$$\rho_1(t,\sigma,c) = (\rho_{11}, ..., \rho_{1n_2}, \sigma_{1n_2+1}, ..., \sigma_{1n_1}).$$

Theorem 1. *If*

1) $\forall (t_0, x_0) \in \Pi_{[0,T]}$ *matrix*

$$\sum_{|k|=2b} A_k(t_0,x_0) \rho_1^k$$

commutes with

$$\sum_{|k|=2b} A_k(t_0,x_0) \int_{\tau}^{t} \rho_1^k(\beta,\sigma,c) d\beta,$$

2) coefficients $A_k(t,x)$ limited and continuous by t, x; evenly continuous over t at $|k| = 2b$, satisfy Gelder's condition on x with an indicator $0 < \alpha \leq 1$ evenly on t, then there is fundamental matrices of the solutions of the Cauchy problem (1)–(2)

$$\Gamma(t,x;\tau,\xi) = G(t,x;\tau,\xi;\xi) + \int\limits_{\tau}^{t} d\beta \int\limits_{R^{n_0}} G(t,x;\beta,\gamma,\gamma)\varphi(\beta,\gamma;\tau,\xi)d\xi,$$

where $G(t,x;\tau,\xi,\xi)$ -is fundamental matrices of the solutions of the Cauchy problem systems with frozen coefficients, φ-desired function with $L\Gamma = 0$.

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On Tauberian conditions for convergence of abstract and trigonometric series

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Let X be a real or complex linear normed space of arbitrary dimension, and $\{x_k\}$ – vectors from X.

It is well-known that $(C,1)$ – summability of the series $\sum x_k$ is the necessary condition for it to converge. However, this condition is not sufficient.

Theorem 1. *Let the series*

$$\sum_{k=0}^{\infty} x_k$$

be $(C,1)$ – summable to $S \in X$ and for some $p \geq 1$ the condition be satisfied

$$(A_p) \qquad \sum_{k=n+1}^{\infty} \|x_k\|^p = O(n^{1-p}), \quad n \to \infty.$$

Then this series converges to S in space X.

Note that the condition (A_p) is weaker than each of the following conditions:

$$\|x_n\| = O\left(\frac{1}{n}\right), \quad n \to \infty; \qquad \text{or} \qquad \sum_{n=1}^{\infty} n^{p-1}\|x_n\|^p < \infty.$$

Using Theorem 1 and some other results, we can prove the following statements about the convergence of the Fourier series

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{1}$$

of the function $f \in L^1(\mathbb{T})$.

Theorem 2. *Let the function f belong to some homogeneous function Banach space X on $\mathbb{T}$ and the following conditions be satisfied*

$$\|e^{int}\|_X = O(1), \quad |n| \to \infty,$$
$$\sum_{|k|\geq n} |c_k|^p = O(n^{1-p}), n \to \infty, \quad p \geq 1. \tag{2}$$

Then the Fourier series (1) converges to f in the Banach space X.

Note that function spaces $L^p(\mathbb{T})$, $1 \leq p < \infty$, separable Orlicz spaces, separable symmetric spaces, space $C(\mathbb{T})$, and real Hardy space are homogeneous.

Theorem 3. *If the coefficients of the Fourier series (1) satisfy the Tauberian condition (2), then the series (1) converges to f uniformly on each compact set of its points of continuity.*

Approximation by parametric extension of Szász-Mirakjan-Kantorovich operators involving the Appell polynomials

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The purpose of this article is to introduce a Kantorovich variant of Szász-Mirakjan-operators by including the Dunkl analogue involving the Appell polynomials, namely the Szász-Mirakjan-Jakimovski-Leviatan type positive linear operators. We study the global approximation in terms of uniform modulus of smoothness and calculate the local direct theorems of the rate of convergence with the help of Lipschitz-type maximal functions in weighted space. Furthermore, the Voronovskaja-type approximation theorems of this new operator are also presented.

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Representation functions of two variables by bicontinued fractions

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Let $u(z,w) = f(z)h(w)$ be a function defined on the compact set $\mathbf{K} \subset \mathbb{C}^2$. The problem of representation of functions of this class by the product of two continued fractions, which is called a bicontinued fraction studied. Some properties of Thiele reciprocal derivatives, Thiele continued fractions and regular C–fractions are proved. The possibility of representation of functions of this class by bicontinued fractions is shown.

The representation by bicontinued fractions of functions $u_1(z,w) = (\delta+\beta z)^\alpha \times \operatorname{tg}(\varepsilon+\gamma w)$, where $\alpha \in \mathbb{C}\backslash\{\mathbb{Z}\}$, $\beta, \gamma, \delta, \varepsilon \in \mathbb{C}\backslash\{0\}$ and $u_2(z,w) = e^{\alpha z} \ln(\beta + \gamma w)$, where $\alpha, \beta, \gamma \in \mathbb{C}\backslash\{0\}$ are considered as an illustration.

The representation of function u_1 by bicontinued fraction is convergent to the function in the domain

$$\mathbb{G}_1 = \{z \in \mathbb{C} : |\arg(\delta + \beta z) - \arg(\delta + \beta z_*)| < \pi\} \times \mathbb{C}\backslash\{\gamma w \neq \tfrac{k\pi}{2} - \varepsilon, k \in \mathbb{Z}\}$$

and bicontinued fraction converges uniformly on an arbitrary compact $\mathbf{K} \subset \mathbb{G}_1$. Similarly, in the region

$$\mathbb{G}_2 = \mathbb{C} \times \{w : w \in \mathbb{C}\backslash\{-\beta/\gamma\}, \big|\arg(\beta + \gamma w) - \arg(\beta + \gamma w_*)\big| < \pi\}$$

constructed bicontinued fraction is convergent to the function u_2 and bicontinued fraction will converge uniformly on an arbitrary compact $\mathbf{K} \subset \mathbb{G}_2$.

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Continuous functions recovery using the noisy Fourier coefficients

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We present the results on a recovery of functions from the classes that are defined in terms of generalized smoothness, from their Fourier coefficients blurred by noise. In a more general statement, the optimal recovery problem for the classes of smooth and analytic functions defined on various compact manifolds, is considered in the classical paper by G. G. Magaril-Il'yaev and K. Y. Osipenko [1].

Let $L_p := L_p([0,1])$, $1 \le p < \infty$, and $C := C([0,1])$ be the spaces of real-valued summable with pth power, and, respectively, continuous on the segment $[0,1]$ functions $f\colon\ [0,1] \to \mathbb{R}$; l_p, $1 \le p < \infty$, be the set of sequences $\xi = (\xi_k)_{k=1}^\infty$ of real numbers. The spaces are equipped with the natural norms.

We estimate from above the quantity

$$\Delta(W^\psi_p, \Lambda, l_p) := \sup_{y \in W^\psi_p} \sup_{\|\xi\|_{l_p} \le 1} \left\| y - \sum_{k=1}^n \lambda^n_k (y_k + \delta \xi_k) \varphi_k \right\|_C, \ 1 \le p < \infty,\ \delta \in (0,1),$$

for functions from the class W^ψ_p, where y_k are the Fourier coefficients of y with respect to some complete orthonormal in the space L_2 system $\Phi = \{\varphi_k\}_{k=1}^\infty$, $y^\delta_k = y_k + \delta\xi_k$, $k = 1, 2, \dots$, are the noisy Fourier coefficients, and the elements of triangular number matrix $\Lambda := \{\lambda^n_k\}_{k=1}^n$, $n = n(\delta) \in \mathbb{N}$, satisfy some additional conditions.

The smoothness parameter $\{\psi(k), k \in \mathbb{N}\}$ of functions from the classes W^ψ_p belongs to the set Ψ_{γ_1,γ_2}, $0 < \gamma_1 < \gamma_2$ of continuous, positive and strictly increasing on $[1,\infty)$ real-valued functions that satisfy the conditions:

1) for some γ, $\gamma_1 < \gamma < \gamma_2$, the function $\phi_-(\tau) := \frac{\tau^\gamma}{\psi(\tau)}$ does not increase for $\tau \geq 1$, and $\phi_-(\tau) \to 0$ as $\tau \to \infty$;

2) the function $\phi_+(\tau) := \frac{\tau^{\gamma_2}}{\psi(\tau)}$ does not decrease for $\tau \geq 1$.

To the set Ψ_{γ_1,γ_2} belong, in particular, power functions $\psi(\tau) = \tau^{\alpha_1}$ with $\gamma_1 < \alpha_1 \leq \gamma_2$, and also functions of the form

$$\psi(\tau) = \tau^{\alpha_1} \left(\log^+(\tau)\right)^{\alpha_2}, \quad \gamma_1 < \alpha_1 < \gamma_2, \quad \alpha_2 \in \mathbb{R}$$

or

$$\psi(\tau) = \tau^{\gamma_2} \left(\log^+(\tau)\right)^{\alpha}, \quad \alpha < 0,$$

where $\log^+(\tau) = \max\{1, \log(\tau)\}$, and the symbol log denotes a logarithm with arbitrary base $a > 0$, $a \neq 1$.

The corresponding results can be found in [2].

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Estimates for the entropy numbers of the Nikol'skii–Besov classes in the space of quasi-continuous functions

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We consider the approximations of the Nikol'skii–Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ (see, for example, [1]) of periodic functions of many variables and establish the estimates for the entropy numbers to these classes in the metric of the QC-space of quasi-continuous functions [2].

Let $\mathbb{R}^d$, $d \geq 1$, be the d-dimensional Euclidean space with the elements $\boldsymbol{x} = (x_1, \ldots, x_d)$ and $(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + \ldots + x_d y_d$. Denote by

$$L_p(\pi_d), \quad \pi_d = \prod_{j=1}^{d} [0, 2\pi], \quad 1 \leq p \leq \infty,$$

the space of functions $f(\boldsymbol{x})$ that are 2π-periodic in each variable, with a standard finite norm.

Let $\mathscr{X}$ be a Banach space, and let

$$B_{\mathscr{X}}(\boldsymbol{y}, r) = \left\{x \in \mathscr{X} \colon \|\boldsymbol{x} - \boldsymbol{y}\| \leq r\right\}$$

be a ball with radius r centered at the point $\boldsymbol{y}$. For the compact set $A \subset \mathscr{X}$ and for $\varepsilon > 0$ the quantity

$$\varepsilon_k(A, \mathscr{X}) = \inf \left\{\varepsilon \colon \exists \boldsymbol{y}^1, \ldots, \boldsymbol{y}^{2^k} \in \mathscr{X} \colon A \subseteq \bigcup_{j=1}^{2^k} B_{\mathscr{X}}(\boldsymbol{y}^j, \varepsilon)\right\}$$

is called the entropy number of the set A with respect to the space $\mathscr{X}$, where $k \in \mathbb{N}$.

Further, let μ be the normed Lebesgue measure on the unit circle. Given a function $f \in L_1(d\mu)$ with the Fourier series

$$f \sim \sum_{s=0}^{\infty} \delta_s(f, x), \quad \delta_0(f, x) = \int_0^{2\pi} f d\mu, \quad \delta_s(f, x) = \sum_{2^{s-1} \leq |k| < 2^s} \widehat{f}(k) e^{ikx}, \ s = 1, 2, \ldots,$$

let us introduce the value
$$\|f\|_{QC} \equiv \int_0^1 \Big\| \sum_{s=0}^{\infty} r_s(\omega)\delta_s(f,x) \Big\|_{L_\infty(d\mu)} d\omega,$$
where $\{r_s(\omega)\}_{s=0}^{\infty}$ is the Rademacher system. By the space of quasi-continuous functions (denote QC) we mean the closure of the set of trigonometric polynomials with respect to this norm.

Spaces of quasi-continuous functions can also be defined in the multidimensional case ($d \geq 2$):
$$\|f\|_{QC} \equiv \big\| \|f(\cdot, \boldsymbol{x}^1)\|_{QC} \big\|_{\infty},$$
where, for $\boldsymbol{x} = (x_1, \ldots, x_d) \in \pi_d$, we set by definition $\boldsymbol{x}^1 = (x_2, \ldots, x_d) \in \pi_{d-1}$. In other words, the QC-norm in these quantity is with respect to the variable x_1 and the sup-norm with respect to the remaining.

We assume that the vector $\boldsymbol{r}$, which is included in the definition of classes $B^{\boldsymbol{r}}_{p,\theta}$ has the form $\boldsymbol{r} = (r_1, \ldots, r_1) \in \mathbb{R}^d_+$.

Theorem 1. *Let*
$$1 < p \leq \infty, \quad 1 \leq \theta < \infty, \quad r_1 > \max\left\{\frac{1}{p}, \frac{1}{2}\right\}.$$
Then for $d \geq 2$ the following relation holds:
$$\varepsilon_M\big(B^{\boldsymbol{r}}_{p,\theta}, QC\big) \ll M^{-r_1}(\log^{d-1} M)^{r_1 + (\frac{1}{p^*} - \frac{1}{\theta})_+} \sqrt{\log M},$$
where $p^* = \min\{p, 2\}$, $a_+ = \max\{a, 0\}$.

Theorem 2. *Let $2 \leq p \leq \infty$, $2 \leq \theta < \infty$, $r_1 > 1/2$. Then for $d \geq 2$ the following relation holds:*
$$\varepsilon_M\big(B^{\boldsymbol{r}}_{p,\theta}, QC\big) \asymp M^{-r_1}(\log^{d-1} M)^{r_1 + \frac{1}{2} - \frac{1}{\theta}} \sqrt{\log M}.$$

The obtained order-estimates for the entropy numbers of the Nikol'skii–Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ in the space QC complement the corresponding results for the Sobolev classes $W^{\boldsymbol{r}}_{p,\alpha}$ and Nikol'skii classes $H^{\boldsymbol{r}}_p$, $1 < p \leq \infty$, which were established by B. S. Kashin and V. N. Temlyakov [2].

Note that the exact-order estimates for the entropy numbers of the Nikol'skii–Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ in the space L_∞ were established only in the two-dimensional case $d = 2$ [3].

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Approximation with regard to the multiple Haar basis

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The talk is based on the author's papers [1–5]. We discuss the results on the approximative properties of one of the multiple Haar basis in the Lebesque spaces $L_q(\mathbb{I}^d)$ of d variable functions defined on the cube $\mathbb{I}^d := [0,1]^d$, $d \geqslant 2$. Namely, about the basis H^d, that is built from the one dimensional Haar basis and the system of characteristic functions of dyadic partition of the segment $[0, 1]$.

The main points of the talk are:

- a description of isotropic Besov spaces and Hölder spaces in terms of conditions on the Fourier–Haar coefficients of elements from these spaces;
- inverse approximation theorems for polynomials, built with respect to the system H^d;
- nonlinear approximation with respect to the basis H^d of functions, that belong to the unit balls of Besov and Hölder spaces;
- open questions concerning the basis H^d in approximation problems.

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Voronovskaya-type theorem for Fejér means of bounded harmonic functions

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Let

$$f(z) = \sum_{k\in\mathbb{Z}} c_k \rho^{|k|} e^{\mathrm{i}kx},$$

where $c_k \in \mathbb{C},\ c_{-k} = \overline{c_k},\ z = \rho e^{\mathrm{i}x}$, be a bounded real-valued harmonic function in the unit disk

$$\mathbb{D} := \{z \in \mathbb{C} :\ |z| < 1\}$$

and let

$$\sigma_n(f)(z) = n^{-1} \sum_{|k|\le n-1} (n - |k|)\, c_k \rho^{|k|} e^{\mathrm{i}kx}, \quad n \in \mathbb{N}, \quad z = \rho e^{\mathrm{i}x}$$

be its Fejér means.

The Voronovskaya-type theorem for the Fejér means $\sigma_n(f)$ states that

$$\lim_{n\to\infty} \left\| n(f - \sigma_n(f)) - \widetilde{f}' \right\| = 0,$$

provided $\left\|\widetilde{f}'\right\| < \infty$, where

$$\widetilde{f}'(z) := \sum_{k\in\mathbb{Z}\setminus\{0\}} |k| c_k \rho^{|k|} e^{\mathrm{i}kx} \quad \text{and} \quad \|f\| := \sup_{z\in\mathbb{D}} |f(z)|.$$

Set

$$hB := \{f : \text{harmonic in } \mathbb{D} \text{ and } \|f\| \le 1\} \quad \text{and} \quad \widetilde{hB} := \{f : \widetilde{f} \in hB\}.$$

Our main result is the following.

Theorem. *Suppose that $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}$ and let $\mathfrak{H}$ be one of the classes hB or $\widetilde{hB}$. Then*

$$\max_{f\in\mathfrak{H}} \left| n(f(z) - \sigma_n(f)(z)) - \widetilde{f}'(z) \right| = \frac{4|z|^{n+1}}{\pi(1-|z|^2)}.$$

For given $z \neq 0$ this maximum is attained only for the function

$$f(t) = e^{i\alpha} \begin{cases} \dfrac{2}{\pi} \arg \dfrac{1 + ib_n(t)e^{-i(n+1)\arg z}}{1 - ib_n(t)e^{-i(n+1)\arg z}}, & \text{if } \mathfrak{H} = hB, \\ \dfrac{2}{\pi} \ln \left| \dfrac{1 + b_n(t)e^{-i(n+1)\arg z}}{1 - b_n(t)e^{-i(n+1)\arg z}} \right|, & \text{if } \mathfrak{H} = \widetilde{hB}, \end{cases}$$

where $\alpha \in \mathbb{R}$ and $b_n(t) := t^n(t-z)/(1-t\overline{z})$.

Corollary. *Let $hB^1 := \{f \in hB : \|f'\| \leq 1\}$. Then the following asymptotic equality holds as $n \to \infty$:*

$$\sup_{f\in hB^1} |f(z) - \sigma_n(f)(z)| = \begin{cases} \dfrac{2}{\pi n} \log \dfrac{1+|z|}{1-|z|} + O\left(\dfrac{|z|^{n+1}}{n^2(1-|z|^2)} \right), & \text{if } z \in \mathbb{D}, \\ \dfrac{2}{\pi n} \log n + O\left(\dfrac{1}{n}\right), & \text{if } z \in \mathbb{T}. \end{cases}$$

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Approximations of functions of classes $C^{\psi}_{\beta,p}$ by Zygmund sums

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Denote by $C^{\psi}_{\beta,p}$, $1 \leq p < \infty$, $\psi(k) \in \mathbb{R}$, $\beta \in \mathbb{R}$, the set of all continuous $2\pi-$ periodic functions f, representable as the convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \Psi_\beta(t)\varphi(x-t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \perp 1, \quad \|\varphi\|_p \leq 1,$$

with a fixed kernel $\Psi_\beta(t) \in L_{p'}$, $1/p + 1/p' = 1$, whose Fourier series has the form

$$\Psi_\beta(t) \sim \sum_{k=1}^{\infty} \psi(k) \cos\left(kt - \frac{\beta\pi}{2}\right).$$

We consider the approximation characteristic

$$\mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C = \sup_{f\in C^{\psi}_{\beta,p}} \|f - Z^s_{n-1}(f)\|_C,$$

where

$$Z^s_{n-1}(f;t) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left(1 - \left(\frac{k}{n}\right)^s\right)(a_k \cos kt + b_k \sin kt), \quad s > 0,$$

is the Zygmund sum of the order $n-1$ of the function $f \in C$, and the best uniform approximations of the classes $C^{\psi}_{\beta,p}$ by trigonometric polynomials t_{n-1} of the order $n-1$, that is, quantities of the form

$$E_n(C^{\psi}_{\beta,p})_C = \sup_{f\in C^{\psi}_{\beta,p}} \inf_{t_{n-1}} \|f(\cdot) - t_{n-1}(\cdot)\|_C.$$

Denote by $\mathfrak{M}_0$ the set of continuous convex downwards functions $g : [1, \infty) \to R_+$ such that

$$\lim_{t\to\infty} g(t) = 0$$

and there exists a constant $K > 0$, such that for all $t \geq 1$

$$\alpha(g;t) := \frac{g(t)}{t|g'(t)|} \geq K > 0, \quad g'(t) := g'(t+0).$$

For a non-negative sequence $g = \{g_k\}_{k=1}^{\infty}$ we write $g \in GM^+$, if $\exists A \geq 1\ \forall n_1,\ n_2 \in \mathbb{N},\ n_1 \leq n_2$:

$$g_{n_1} + \sum_{k=n_1}^{m-1} |g_k - g_{k+1}| \leq A g_m, \quad m = \overline{n_1, n_2},$$

and $g \in GA^+$ if $\exists \varepsilon > 0\ \exists K > 0\ \forall n_1, n_2 \in \mathbb{N},\ n_1 \leq n_2$: $g_{n_1} n_1^{-\varepsilon} \leq K g_{n_2} n_2^{-\varepsilon}$.

Let $\delta > 0$ $g_\delta(t) := \psi(t) t^\delta$, $t \in [1, \infty)$.

Theorem. *Let $s > 0$, $1 \leq p < \infty$, $g_{1/p} \in \mathfrak{M}_0$, $g_{s+1/p} \in GM^+ \cap GA^+$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for $1 < p < \infty$, if*

$$\sum_{k=n}^{\infty} \psi^{p'}(k) k^{p'-2} < \infty \quad \text{and} \quad \inf_{t\geq 1} \alpha(g_{1/p}; t) > \frac{p'}{2},$$

the following order estimates are true

$$E_n\Big(C^{\psi}_{\beta,p}\Big)_C \asymp \mathcal{E}\Big(C^{\psi}_{\beta,p}; Z^s_{n-1}\Big)_C \asymp \Big(\sum_{k=n}^{\infty} \psi^{p'}(k) k^{p'-2}\Big)^{1/p'}, \ 1/p + 1/p' = 1.$$

Then for $p = 1$, if

$$\sum_{k=n}^{\infty} \psi(k) < \infty \quad \text{and} \quad \inf_{t\geq 1} \alpha(g_1; t) > 1,$$

the following order estimates are true

$$E_n\left(C^{\psi}_{\beta,1}\right)_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,1}; Z^s_{n-1}\right)_C \asymp \begin{cases} \sum\limits_{k=n}^{\infty} \psi(k), & \cos\frac{\beta\pi}{2} \neq 0; \\ \psi(n) n, & \cos\frac{\beta\pi}{2} = 0. \end{cases}$$

Jackson-type inequalities for almost periodic functions in the Besicovitch-Stepanets spaces

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Let $f(x)$ be an arbitrary almost periodic complex-valued Besicovitch function of the class $B = B_1$ (B-a.p. function), whose Fourier exponents have a single limit point at infinity [1]. Let us write its Fourier series in the symmetric form

$$\sum_{k\in\mathbb{Z}} A_k \mathrm{e}^{\mathrm{i}\lambda_k x},$$

where $\lambda_0 := 0$, $\lambda_{-k} = -\lambda_k$, $|A_k| + |A_{-k}| > 0$, $\lambda_{k+1} > \lambda_k > 0$, $k > 0$.

Developing the ideas of Stepanets [2, Ch.9], for a fixed $1 \leq p < \infty$, we introduce into consideration the space $B\mathcal{S}^p$ of all functions $f \in B$-a.p. with the finite norm

$$\|f\|_p := \|f\|_{B\mathcal{S}^p} = \bigg(\sum_k |A_{\lambda_k}(f)|^p\bigg)^{1/p}.$$

In this case, functions with the same Fourier series are identified in $B\mathcal{S}^p$.

Let us set

$$E_{\lambda_n}(f)_p = \inf_{g\in G_{\lambda_n}} \|f-g\|_p, \quad n=1,2,\ldots,$$

where G_{λ_n} is a subset of functions of $B\mathcal{S}^p$ such that their spectrum belongs to the interval $(-\lambda_n,\lambda_n)$. Let Φ be the set of continuous bounded nonnegative pair functions φ such that $\varphi(0)=0$ and $\mathrm{mes}\{t\in\mathbb{R}: \varphi(t)=0\}=0$. For a fixed $\varphi\in\Phi$, define the generalized modulus of smoothness of the function $f\in B\mathcal{S}^p$ by

$$\omega_\varphi(f,\delta)_p := \omega_\varphi(f,\delta)_{B\mathcal{S}^p} = \sup_{|h|\le\delta}\left(\sum_{k\in\mathbb{Z}}\varphi^p(\lambda_k h)|A_k(f)|^p\right)^{1/p}, \quad \delta\ge 0.$$

Let M be the set of functions μ, non-decreasing bounded and non-constant on $[0,\tau]$.

Theorem. *Assume that $f\in B\mathcal{S}^p$, $1\le p<\infty$. Then for $\varphi\in\Phi$, $\tau>0$ and $n\in\mathbb{N}$*

$$E_{\lambda_n}(f)_p \le C_{n,\varphi,p}(\tau)\omega_\varphi\Big(f,\frac{\tau}{\lambda_n}\Big)_p, \tag{1}$$

where

$$C^p_{n,\varphi,p}(\tau) = \inf_{\mu\in M(\tau)}\frac{\mu(\tau)-\mu(0)}{I_{n,\varphi,p}(\tau,\mu)}, \quad I_{n,\varphi,p}(\tau,\mu) = \inf_{k\in\mathbb{N}:\, k\ge n}\int_0^\tau \varphi^p\Big(\frac{\lambda_k t}{\lambda_n}\Big)\mathrm{d}\mu(t). \tag{2}$$

Moreover, there is a function $\mu_*\in M(\tau)$ that realizes the corresponding exact lower bound in (2). Inequality (1) is unimprovable on the set of all functions $f\in B\mathcal{S}^p$, $f\not\equiv\mathrm{const}$, in the sence that for any $\varphi\in\Phi$ and $n\in\mathbb{N}$

$$\sup\left\{\frac{E_{\lambda_n}(f)_p}{\omega_\varphi(f,\frac{\tau}{\lambda_n})_p} \ :\ f\in B\mathcal{S}^p,\ \ f\not\equiv\mathrm{const}\right\} = C_{n,\varphi,p}(\tau).$$

For 2π-periodic functions from the Stepanets spaces $\mathcal{S}^p$, unimprovable Jackson-type inequalities were established in [3], formulated in terms of the best approximation of functions by trigonometric polynomials and classical moduli of smoothness.

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On asymptotic estimates for the widths of classes of functions of high smothness

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Denote by $W^r_{\bar\beta,2}$ the set of all 2π-periodic functions f, representable as convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x-t)B_{r,\bar\beta}(t)dt, \quad a_0\in\mathbb{R}, \tag{1}$$

the fixed kernel $B_{r,\bar\beta}(\cdot)$ of the form:

$$B_{r,\bar\beta}(t) = \sum_{k=1}^{\infty} k^{-r}\cos\left(kt-\frac{\beta_k\pi}{2}\right), \quad r>\frac{1}{2}, \quad \beta_k\in\mathbb{R},$$

with functions

$$\varphi\in B^0_2 = \{g\in L_2:\ \|g\|_{L_2}\le 1,\ g\perp 1\}.$$

If $r \in \mathbb{N}$ and $\beta_k \equiv r$ then the functions $B_{r,\beta}(\cdot)$ are the well-known Bernoulli kernels and the classes $W^r_{\bar\beta,2}$ coincide with the well-known classes W^r_2 which consist of 2π-periodic functions f with absolutely continuous derivatives $f^{(k)}$ up to $(r-1)$-th order inclusive and such that $\|f^{(r)}\|_{L_2} \le 1$. In addition, for almost everywhere $x \in \mathbb{R}$

$$f^{(r)}(x) = f^r_r(x) = \varphi(x),$$

where φ is the function from (1).

Further, let K be a convex centrally symmetric subset of C and let B be a unit ball of the space C. Let also F_N be an arbitrary N-dimensional subspace of space C, $N \in \mathbb{N}$, and $\mathscr{L}(C, F_N)$ be a set of linear operators from C to F_N. By $\mathscr{P}(C, F_N)$ denote the subset of projection operators of the set $\mathscr{L}(C, F_N)$, that is, the set of the operators A of linear projection onto the set F_N such that $Af = f$ when $f \in F_N$. The quantities

$$b_N(K, C) = \sup_{F_{N+1}} \sup\{\varepsilon > 0 : \varepsilon B \cap F_{N+1} \subset K\},$$

$$d_N(K, C) = \inf_{F_N} \sup_{f\in K} \inf_{u\in F_N} \|f - u\|_C,$$

$$\lambda_N(K, C) = \inf_{F_N} \inf_{A\in\mathscr{L}(C,F_N)} \sup_{f\in K} \|f - Af\|_C,$$

$$\pi_N(K, C) = \inf_{F_N} \inf_{A\in\mathscr{P}(C,F_N)} \sup_{f\in K} \|f - Af\|_C,$$

are called Bernstein, Kolmogorov, linear, and projection N-widths of the set K in the space C, respectively.

Theorem. *Let $r > 1$, $n \in \mathbb{N}$, $\bar\beta = \{\beta_k\}_{k=1}^\infty, \beta_k \in \mathbb{R}$ and*

$$\lim_{n\to\infty} \frac{r}{n} = \infty.$$

Then the following asymptotic equalities hold

$$\left.\begin{array}{l} P_{2n}(W^r_{\bar\beta,2}, C) \\ P_{2n-1}(W^r_{\bar\beta,2}, C) \end{array}\right\} = n^{-r}\left(\frac{1}{\sqrt{\pi}} + \mathcal{O}(1)\left(1 + \frac{1}{n}\right)^{-r}\right), \tag{2}$$

where P_N is any of the widths b_N, d_N, λ_N or π_N, and $\mathcal{O}(1)$ are the quantities uniformly bounded in all parameters.

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Asymptotically best possible Lebesque-type inequalities on the classes $C^{\alpha,r}_{\beta}C$

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For the functions f, which can be represented in the form of the convolution

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\left(kt - \frac{\beta\pi}{2}\right) \varphi(x - t)dt,$$

$\varphi \in C$, $\varphi \perp 1$, $\alpha > 0$, $r \in (0,1)$, $a_0, \beta \in \mathbb{R}$, we establish the Lebesgue-type inequalities of the form

$$\|f - S_{n-1}(f)\|_C \le e^{-\alpha n^r} \left(\frac{4}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \gamma_n \right) E_n(\varphi)_C, \ |\gamma_n| \le 20\pi^4.$$

These Lebesgue inequalities take place for all numbers n that are larger than some number $n_1 = n_1(\alpha, r)$, which constructively defined via parameters α and r. We prove that there exists a function, such that the sign "$\le$" in given estimate can be changed for "=".

Direct and inverse theorems of approximation of functions given on hexagonal domains by the Taylor-Abel-Poisson means

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We present the results of our joint work with professor Jürgen Prestin (University of Lübeck, Germany) and professor Viktor Savchuk (Institute of Mathematics of NASU). The approximative properties of the Taylor-Abel-Poisson linear summation method of Fourier series are considered for functions of several variables, periodic with respect to the hexagonal lattice. This type of periodicity is defined by the hexagon lattice given by $\mathcal{H}\mathbb{Z}^2$ (see, for example [1, 2]), where

$$\mathcal{H} = \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}, \quad \Omega_{\mathcal{H}} = \left\{ (x_1, x_2) : \ -1 \le x_2, \frac{\sqrt{3}}{2} x_1 \pm \frac{1}{2} x_2 < 1 \right\}.$$

Use the homogeneous coordinates $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3$ such that $t_1 + t_2 + t_3 = 0$ and write $\mathbf{t} \in \mathbb{R}^3_{\mathcal{H}}$. If we set

$$t_1 = -\frac{x_2}{2} + \frac{\sqrt{3} x_1}{2}, \quad t_2 = x_2, \quad t_3 := -\frac{x_2}{2} - \frac{\sqrt{3} x_1}{2},$$

then the hexagonal domain $\Omega_{\mathcal{H}}$ becomes

$$\Omega = \{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3_{\mathcal{H}} : \ -1 \le t_1, t_2, t_3 < 1\},$$

which is the intersection of the plane $t_1 + t_2 + t_3 = 0$ with the cube $[-1, 1]^3$.

A function f is called periodic with respect to the hexagonal lattice $\mathcal{H}$ (or $\mathcal{H}$-periodic) if $f(x) = f(x + \mathcal{H}k)$, $k \in \mathbb{Z}^2$. In homogeneous coordinates, a function $f(\mathbf{t})$ is $\mathcal{H}$-periodic if $f(\mathbf{t}) = f(\mathbf{t} + \mathbf{j})$ whenever $\mathbf{j} \equiv \mathbf{0} \pmod 3$.

Let $L_p = L_p(\Omega)$, $1 \le p \le \infty$, be the space of all functions f, given on the hexagonal domain Ω, with the usual norm

$$\|f\|_p := \begin{cases} \left(\frac{1}{|\Omega|} \int_\Omega |f(\mathbf{t})|^p \mathrm{d}\mathbf{t} \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{\mathbf{t} \in \Omega} |f(\mathbf{t})|, & p = \infty. \end{cases}$$

The inner product is defined by

$$\langle f, g \rangle = \frac{1}{|\Omega|} \int_\Omega f(\mathbf{t}) \overline{g(\mathbf{t})} \mathrm{d}\mathbf{t}.$$

The set

$$\{\phi_{\mathbf{j}}(\mathbf{t}) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3} \mathbf{k} \cdot \mathbf{t}} : \ \mathbf{j} \in \mathbb{Z}^3_{\mathcal{H}}\}$$

is an orthonormal basis of $L_2(\Omega)$ [1], and for any function $f \in L_1(\Omega)$ the Fourier series on the system ϕ has the form

$$S[f](\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{Z}^3_{\mathcal{H}}} \widehat{f}(\mathbf{k}) \phi_{\mathbf{k}}(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{Z}^3_{\mathcal{H}}} \widehat{f}(\mathbf{k}) \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}(k_1 t_2 + k_2 t_2 + k_3 t_3)}, \quad \widehat{f}(\mathbf{k}) := \langle f, \phi_{\mathbf{k}} \rangle.$$

Set

$$\mathbb{J}_\nu := \{\mathbf{k} \in \mathbb{Z}^3_{\mathcal{H}} : |\mathbf{k}| := \max_j \{|k_j|\} = \nu\}, \quad \nu = 0, 1, \ldots,$$

and for any $\varrho \in [0,1)$ and $r \in \mathbb{N}$, consider the transformation

$$A_{\varrho,r}(f)(\mathbf{t}) := \sum_{\nu=0}^{\infty} \lambda_{\nu,r}(\varrho) \sum_{\mathbf{k}\in\mathbb{J}_\nu} \widehat{f}(\mathbf{k})\phi_{\mathbf{k}}(\mathbf{t}),$$

where for $\nu = 0, 1, \ldots, r-1$, the coefficients are defined by $\lambda_{\nu,r}(\varrho) \equiv 1$ and

$$\lambda_{\nu,r}(\varrho) := \sum_{j=0}^{r-1} \binom{\nu}{j} (1-\varrho)^j \varrho^{\nu-j} = \sum_{j=0}^{r-1} \frac{(1-\varrho)^j}{j!} \frac{\mathrm{d}^j}{\mathrm{d}\varrho^j} \varrho^\nu, \quad \nu = r, r+1, \ldots$$

If for a function $f \in L_1(\Omega)$ and $n \in \mathbb{N}$, there exists the function $g \in L_1(\Omega)$ such that $\widehat{g}(\mathbf{k}) = 0$ when $|\mathbf{k}| < n$ and

$$\widehat{g}(\mathbf{k}) = \frac{|\mathbf{k}|!}{(|\mathbf{k}|-n)!} \widehat{f}(\mathbf{k})$$

when $|\mathbf{k}| \geq n$, $\mathbf{k} \in \mathbb{Z}^3_{\mathcal{H}}$, then for the function f, there exists the radial derivative $g =: f^{[n]}$ of order n.

In the space $L_p(\Omega)$, the K–functional of a function f generated by the radial derivative of order n, is the following quantity:

$$K_n(\delta, f)_p := \inf \left\{ \|f-h\|_p + \delta^n \left\| h^{[n]} \right\|_p : h^{[n]} \in L_p(\Omega) \right\}, \quad \delta > 0.$$

Let $\mathcal{Z}_n$, $n \in \mathbb{N}$, denote the sets of all continuous strictly increasing functions $\omega(t)$, $t \in [0,1]$, with $\omega(0) = 0$ satisfying the following conditions:

$$\int_0^\delta \frac{\omega(t)}{t} \mathrm{d}t = \mathcal{O}(\omega(\delta)), \quad \delta \to 0+, \quad \text{and} \quad \int_\delta^1 \frac{\omega(t)}{t^{n+1}} \mathrm{d}t = \mathcal{O}\left(\frac{\omega(\delta)}{\delta^n}\right), \quad \delta \to 0+.$$

Theorem. *Assume that*

$$f \in L_p(\Omega), \quad 1 \leq p \leq \infty, \quad n, r \in \mathbb{N}, \quad n \leq r \quad \textit{and} \quad \omega \in \mathcal{Z}_n.$$

Then

$$\|f - A_{\varrho,r}(f)\|_p = \mathcal{O}\left((1-\varrho)^{r-n}\omega(1-\varrho)\right), \quad \varrho \to 1-,$$

iff there exists the derivative $f^{[r-n]} \in L_p(\Omega)$ *and*

$$K_n\left(\delta, f^{[r-n]}\right)_p = \mathcal{O}(\omega(\delta)), \quad \delta \to 0+.$$

For 2π-periodic functions and for functions of several variables 2π-periodic in each variable, similar direct and inverse theorems of approximation by the Taylor-Abel-Poisson means in the integral metrics were given in [3] and [4] respectively.

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P-summability method applied on an integral type operator based on multivariate q-Lagrange polynomials

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In this work, using q-Riemann integral, we introduce an integral type generalization of an operator [1] based on q-multivariate Lagrange polynomials. Using P-summability method, we provide a non-trivial Korovkin type theorem for the proposed operators and establish error in approximation in terms of the modulus of continuity. At last, we discuss an advantage of our convergence theorem over the classical Korovkin's theorem on linear positive operators.

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The Landau-Kolmogorov problem on a finite interval in the Taikov case

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Let $k,r\in\mathbb{Z}_+$, $0\leqslant k\leqslant r-1$. Throughout the note we study classes of functions defined on the interval $[-1,1]$. We consider the Landau-Kolmogorov problem in the Taikov case, *i.e.* the problem of finding the modulus of continuity of operator $D^k: L_2\to L_\infty$ on the class $W_2^r := \left\{f\in L_2^r\,:\,\left\|f^{(r)}\right\|_2\leqslant 1\right\}$:

$$\Omega(\delta)=\Omega(\delta;D^k;W_2^r)=\sup\left\{\|f^{(k)}\|_\infty\,:\,f\in W_2^r,\ \|f\|_2\leqslant\delta\right\},\quad \delta\geqslant 0. \tag{1}$$

Also, we consider its pointwise analogue: for $t\in[-1,1]$, find

$$\Omega_t(\delta)=\Omega(\delta;D_t^k;W_2^r)=\sup\left\{|f^{(k)}(t)|\,:\,f\in W_2^r,\ \|f\|_2\leqslant\delta\right\},\quad \delta\geqslant 0. \tag{2}$$

Problems (1) and (2) are fully solved for functions defined on $\mathbb{R}$ [1], $\mathbb{T}$ [2], $\mathbb{R}_+$ [3, 4]. However, no sharp results are known for functions defined on finite interval.

We solve problem (2) for all admissible k, r and δ. Let us formulate corresponding solution in the case $t=-1$. For $\lambda\geqslant 0$, consider boundary value problem

$$\begin{array}{ll} (-1)^r u^{(2r)}(x)+\lambda u(x)=0, & x\in(-1,1),\\ u^{(s)}(-1)=(-1)^{k-1}\delta_{r-k-1,s}, & s=0,1,\dots,r-1,\\ u^{(s)}(1)=0, & s=0,1,\dots,r-1. \end{array} \tag{3}$$

Theorem. *Let $r\in\mathbb{N}$ and $k\in\mathbb{Z}_+$, $k\leqslant r-1$. Then, for every $\delta>0$,*

$$\Omega_0(\delta):=\Omega(\delta;D_{-1}^k:L_2\to\mathbb{R};W_2^r)=\|u_\lambda^{(r)}\|_2\delta+\|u_\lambda\|_2,$$

where $\lambda=\lambda(\delta)>0$ is such that $\lambda\delta\cdot\|u_\lambda\|_2=\|u_\lambda^{(r)}\|_2$ and $u_\lambda\in L_2^{2r}$ solves (3).

For problem (1) we formulate Karlin's type conjecture: for every $\delta > 0$,

$$\Omega(\delta) = \sup\{\Omega_t(\delta) \,:\, t \in [-1,1]\} = \Omega_{-1}(\delta). \tag{4}$$

We confirm conjecture (4) for $r = 1$ and $r = 2$. Also, we show that conjecture (4) holds true for some $r \in \mathbb{N}$ and $k \in \{0, 1, \ldots, r-1\}$ immediately, if we prove that for every eigen-function φ of the boundary value problem

$$\begin{array}{ll} (-1)^r u^{(2r)}(x) = \lambda u(x) = 0, & x \in (-1,1), \\ u^{(s)}(-1) = u^{(s)}(1) = 0, & s = 0, 1, \ldots, r-1, \end{array}$$

its derivative $\varphi^{(r+k)}$ has the property:

$$\forall x \in [-1,1] : |\varphi^{(r+k)}(x)| \leqslant |\varphi^{(r+k)}(-1)|.$$

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Kantorovich modifications of sampling type operators for multivalued functions

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The present talk deals with a construction of Kantorovich version of sampling type operators in multivariate case. We present pointwise convergence of the series at continuity points of target functions and uniform convergence for uniformly logarithmic continuous functions. A rate of convergence for the series is also presented via logarithmic modulus of continuity and a Voronovskaya theorem for Mellin differentiable functions is obtained as well.

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On the widths of classes of functions of two variables

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By $L_2(\mathbb{R}^2)$, where

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : -\infty < x, y < \infty\},$$

we denote the space of real functions measurable and square summable on the plane $\mathbb{R}^2$. By symbol $L_{2,\gamma}(\mathbb{R})$, where $\gamma(x, y) = \exp(-x^2 - y^2)$, stands for the set of functions $f : \mathbb{R}^2 \to \mathbb{R}$ for which $\gamma^{1/2} \cdot f \in L_2(\mathbb{R}^2)$.

The formula

$$\|f\|_{2,\gamma} = \|f\|_{L_{2,\gamma}(\mathbb{R}^2)} = \left\{ \iint\limits_{\mathbb{R}^2} \gamma(x, y) f^2(x, y) dx dy \right\}^{1/2}$$

defines the norm in the space $L_{2,\gamma}(\mathbb{R}^2)$.

Let's consider the differential operator

$$D := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}.$$

By symbol $L^r_{2,\gamma}(\mathbb{R}^2)$, $r \in \mathbb{N}$, we denote the class of functions $f \in L_{2,\gamma}(\mathbb{R}^2)$, having generalized partial derivatives $\dfrac{\partial^k f}{\partial x^i \partial y^j}$, where $i + j = k, k = \overline{1, 2r}$; $i, j \in \mathbb{Z}_+$, which belong to the space $L_{2,\gamma}(\mathbb{R}^2)$. In this case we consider that

$$D^r f = D(D^{r-1} f), \quad r \in \mathbb{N}, quad D^0 f \equiv f.$$

It is obviously that for any $f \in L^r_{2,\gamma}(\mathbb{R}^2)$ the function $D^r f$ belongs to $L_{2,\gamma}(\mathbb{R}^2)$. Hence, we shall write $L^r_{2,\gamma}(D, \mathbb{R}^2)$ instead of $L^r_{2,\gamma}(\mathbb{R}^2)$.

Let $L^{r,0}_{2,\gamma}(D, \mathbb{R}^2)$, $r \in \mathbb{N}$, be the classes, consisting of functions $f \in L^r_{2,\gamma}(D, \mathbb{R}^2)$ such that them Fourier - Hermite coefficients $c_{i0}(f) = c_{j0}(f) = c_{00}(f) = 0$ for any $i, j \in \mathbb{N}$.

By symbol $\Omega_{m,\gamma}(f, t)$, where $t \in (0, 1)$, $m \in \mathbb{N}$, we denote the generalized modulus of continuity of the m-th order of function $f \in L_{2,\gamma}(\mathbb{R})$, formed by the operator of generalized shift

$$F_h : L_{2,\gamma}(\mathbb{R}^2) \to L_{2,\gamma}(\mathbb{R}^2),$$

which has the form [1]

$$F_h f(x, y) = \frac{1}{\pi} \iint\limits_{\mathbb{R}^2} f\big(x\sqrt{1 - h^2} + hu, y\sqrt{1 - h^2} + hv\big)\gamma(u, v) du dv.$$

Let $\mathcal{P}_{n-1}$, where $n \in \mathbb{N}$, be the subspace of algebraic polynomials of the form

$$p_{n-1}(x, y) = \sum_{\substack{i+j=0 \\ (i,j \in \mathbb{Z}_+)}}^{n-1} a_{ij} x^i y^j$$

and let $E_{n-1}(f)_{2,\gamma}$ be the best approximation of a function $f \in L_{2,\gamma}(\mathbb{R}^2)$ by elements of the subspace $\mathcal{P}_{n-1}$ in $L_{2,\gamma}(\mathbb{R}^2)$. For the set $\mathfrak{N} \subset L_{2,\gamma}(\mathbb{R}^2)$ we take

$$E_{n-1}(\mathfrak{N})_{2,\gamma} = \sup\{E_{n-1}(f)_{2,\gamma} : f \in \mathfrak{N}\}.$$

Let $\Psi(t)$, $t \in [0,1]$, be a monotonically increasing and continuous function such that $\Psi(0) = 0$. By $W_2^{r,0}(\Omega_{m,\gamma}, \Psi)$, where $r, m, \in \mathbb{N}$, we denote the classes consisting of functions $f \in L_{2,\gamma}^{r,0}(D, \mathbb{R}^2)$ for which the inequality

$$\Omega_{m,\gamma}(D^r f, t) \leqslant \Psi(t)$$

holds for any value $t \in (0,1)$.

We formulate one of our results.

Theorem. *Let $m, r \in \mathbb{N}$; $n \in \mathbb{N}\backslash\{1\}$; $k = \overline{0,n}$, and let the function Ψ be a majarant which satisfies the condition*

$$\inf_{0<t<1} \frac{(1+(1-t^2)^{1/2})^m \Psi(t)}{t^{2m}} = 2^m \overline{\lim_{t\to 0+}} \frac{\Psi(t)}{t^{2m}}. \tag{1}$$

Then the following equalities hold

$$p_{n(n+1)/2+k}(W_2^{r,0}(\Omega_{m,\gamma}, \Psi); L_{2,\gamma}(\mathbb{R}^2)) = E_{n-1}(W_2^{r,0}(\Omega_{m,\gamma}, \Psi))_{2,\gamma} = \frac{2^{m-r}}{n^{m+r}} \overline{\lim_{t\to 0+}} \frac{\Psi(t)}{t^{2m}}, \tag{2}$$

where

$$p_{n(n+1)/2+k}(W_2^{r,0}(\Omega_{m,\gamma}, \Psi); L_{2,\gamma}(\mathbb{R}^2))$$

is any of the widths: Bernstein, Kolmogorov, linear, Gel'fand, projective, orthowidth (or Fourier width).

There are different wide classes of majorants which satisfy the condition (1). Let's consider one of them:

$$\Psi_m(\eta, t) = \Big(\frac{(1+t^2)^m - 1}{1+(1-t^2)^{1/2}}\Big)^m \eta(t), \quad 0 \leqslant t \leqslant 1,$$

where $\eta(t)$ is an arbitrary continuous, nondecreasing and positive function on the segment $[0,1]$.

The formula (1) holds for $\Psi_m(\eta)$. Hence, we obtain from (2)

$$p_{n(n+1)/2+k}(W_2^{r,0}(\Omega_{m,\gamma}, \Psi_m(\eta)); L_{2,\gamma}(\mathbb{R}^2)) = E_{n-1}(W_2^{r,0}(\Omega_{m,\gamma}, \Psi_m(\eta)))_{2,\gamma} = \frac{m^m}{2^r n^{m+r}} \eta(0), \quad k = \overline{0,n}.$$

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Approximation by sampling type operators and applications

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In this talk, I will deal with some recent approximation results for sampling type operators, as estimates, convergence, rate of approximation, inverse theorems and saturation order. I will discuss the importance of the family of the operators considered, not only from a theoretical point of view, but also through their use in concrete applications.

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